OPTIMAL DISTRIBUTION OF LIMITED RESOURCE IN A CONTEXT OF UNCERTAINTY

Object of research: methods of rational distribution of limited resource in a context of uncertainty. It is assumed that the system consists of a set of independently functioning elements, and effectiveness of these elements depends on the level of investment. The complexity of solving the problem using traditional methods depends on and is determined by the type of analytical description of production functions of the system elements. Significant difficulties arise if parameters of production functions are uncertain quantities, specified, for example, in terms of fuzzy mathematics. This circumstance emphasizes the relevance of studying approaches to solving the set problem of resource distribution for the case when its parameters are not clearly defined, and this circumstance also determines the goal of the work. Problems arising from the goal consist in developing mathematical models and methods for rational resource distribution for the main types of production functions with their parameters being fuzzy numbers of (L-R) type. To solve these problems, a method is proposed that transforms the original fuzzy problems into clear ones that can be solved using standard constrained optimization technologies. An important result of the research consists in the fact that the proposed method is universal, that is, it is applied in the same way to solve problems of resource distribution in systems wherein production functions of the elements can be of arbitrary nature.

Keywords: rational resource distribution; fuzzy parameters; fuzzy optimization; fuzzy mathematics.

Introduction. The problem of rational distribution of a limited resource belongs to the class of nonlinear mathematical programming problems [1]. Constructional features of such problems determine their belonging to constrained optimization problems [2, 3]. Let us consider the specific problem of distributing a one-dimensional resource in a multi-element production system [4].

Let us introduce a vector $x = (x_1, x_2, ..., x_n)$, that determines the distribution of the resource among the elements of the system, and a set of
\( f_j(x_j) = a_j x_j^\alpha, j = 1,2, ..., n, \) one-parameter production functions of the system elements.

Let us set the multiplicative production function of the system, which determines the criterion for the efficiency of resource distribution

\[
F(x) = \prod_{j=1}^{n} a_j x_j^\alpha = \prod_{j=1}^{n} (a_j \prod_{j=1}^{n} x_j^\alpha) = a_0 \prod_{j=1}^{n} x_j^\alpha. \tag{1}
\]

Let us introduce a limitation on the amount of total resource consumed

\[
\sum_{j=1}^{n} x_j = C. \tag{2}
\]

We are going to find the unknown vector \( X = (x_1, x_2, ..., x_n) \) by means of using the method of Lagrange undetermined multipliers. Let us introduce the Lagrange function

\[
\phi(x, \lambda) = a_0 \prod_{j=1}^{n} x_j^\alpha - \lambda \left( \sum_{j=1}^{n} x_j - C \right). \tag{3}
\]

Next

\[
\frac{\partial \phi(x, \lambda)}{\partial x_{j_0}} = a_0 \alpha \left( \prod_{j \neq j_0} x_j^\alpha \right) x_{j_0}^{\alpha-1} - \lambda = 0 \tag{4}
\]

or

\[
\frac{a_0 \alpha}{x_{j_0}} \prod_{j=1}^{n} x_j^\alpha - \lambda = 0.
\]

Hence

\[
x_{j_0} = \frac{a_0 \alpha}{\lambda} \prod_{j=1}^{n} x_j^\alpha, \quad j_0 = 1,2, ..., n. \tag{5}
\]

We are going to find the unknown value \( \lambda \) from the equation (2). Here we have the following

\[
\sum_{j=1}^{n} x_j = \frac{1}{\alpha} n a_0 \alpha \prod_{j=1}^{n} x_j^\alpha = C,
\]
\[
\frac{1}{\lambda} = \frac{c}{na_0 \alpha} \prod_{j=1}^{n} x_j^\alpha.
\]  

(6)

By means of substituting (6) into (5), we receive the following

\[
x_j = \frac{c}{na_0 \alpha} \prod_{j=1}^{n} x_j^\alpha \cdot \alpha_0 \alpha \prod_{j=1}^{n} x_j^\alpha = \frac{c}{n}.
\]

The triviality of the resulting solution can be explained by the multiplicativity of criterion (1) and the extreme simplicity of the production function. Let us now obtain a solution to this problem by introducing another, alternative option for constructing a criterion. Let it be as follows

\[
F(x) = \sum_{j=1}^{n} f_j(x_j) = \sum_{j=1}^{n} a_j x_j^\alpha.
\]  

(7)

Again, by means of using the method of Lagrange undetermined multipliers, we obtain the following

\[
\phi(x, \lambda) = \sum_{j=1}^{n} a_j x_j^\alpha - \lambda \left( \sum_{j=1}^{n} x_j - C \right).
\]

Next

\[
\frac{\partial \phi(x, \lambda)}{\partial x_j} = a_j \alpha x_j^{\alpha-1} - \lambda = 0,
\]

\[
x_j^{\alpha-1} = \frac{\lambda}{a_j \alpha}, \quad j = 1, 2, \ldots, n,
\]

whence

\[
x_j = \left( \frac{\lambda}{\alpha} \right)^{\frac{1}{\alpha-1}} \left( \frac{1}{a_j} \right)^{\frac{1}{\alpha-1}}.
\]

We are going to find the unknown factor \( \left( \frac{\lambda}{\alpha} \right)^{\frac{1}{\alpha-1}} \) by means of using restrictions (2):

\[
\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \left( \frac{\lambda}{\alpha} \right)^{\frac{1}{\alpha-1}} \left( \frac{1}{a_j} \right)^{\frac{1}{\alpha-1}} =
\]

7
\[
\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} \sum_{j=1}^{n} \left(\frac{1}{a_j}\right)^{\frac{1}{\alpha-1}} = C.
\]

Hence

\[
\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} = \frac{C}{\sum_{j=1}^{n} \left(\frac{1}{a_j}\right)^{\frac{1}{\alpha-1}}}.
\]

Then

\[
x_j = \frac{C \cdot \left(\frac{1}{a_j}\right)^{\frac{1}{\alpha-1}}}{\sum_{j=1}^{n} \left(\frac{1}{a_j}\right)^{\frac{1}{\alpha-1}}} - \frac{C a_j^{\frac{1}{\alpha-\alpha}}}{\sum_{j=1}^{n} a_j^{\frac{1}{1-\alpha}}}.
\]

Let it be, for example, as follows \(\alpha = \frac{1}{2}\). Herewith we receive the following

\[
x_j = C \cdot \frac{a_j^{\frac{1}{1-0.5}}}{\sum_{j=1}^{n} (a_j)^{\frac{1}{1-0.5}}} = C \cdot \frac{a_j^{2}}{\sum_{j=1}^{n} a_j^{2}}, j = 1, 2, ..., n.
\]

Let us now solve this problem using a more informative analytical representation of the production function of the system elements and an additive criterion for its efficiency.

Let us introduce the following

\[
f_j(x) = a_j x_j^{\alpha_j}.
\]

Next

\[
F(x) = \sum_{j=1}^{n} f_j(x_j) = \sum_{j=1}^{n} a_j x_j^{\alpha_j}.
\]

In this case the Lagrange function has the following form:
\[
\phi(x, \lambda) = \sum_{j=1}^{n} a_j x_j^{\alpha_j} - \lambda \left( \sum_{j=1}^{n} x_j - C \right).
\]

Next
\[
\frac{\partial \phi(x, \lambda)}{\partial x_j} = a_j \alpha_j x_j^{\alpha_j-1} - \lambda = 0, \quad j = 1,2, ..., n. \quad (9)
\]

Hence
\[
x_j^{\alpha_j-1} = \frac{\lambda}{a_j \alpha_j}, \quad x_j = \left( \frac{\lambda}{a_j \alpha_j} \right)^{\frac{1}{\alpha_j-1}} = \\
= (\lambda)^{\frac{1}{\alpha_j-1}} \cdot \left( \frac{1}{a_j \alpha_j} \right)^{\frac{1}{\alpha_j-1}}.
\]

Let us find the factor \((\lambda)^{\frac{1}{\alpha_j-1}}\), by means of using (2) again.

\[
\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \left( \frac{\lambda}{a_j \alpha_j} \right)^{\frac{1}{\alpha_j-1}} = C.
\]

The resulting equation (being non-linear relative to \(\lambda\)) can be solved numerically. Its simple analytical solution can be found in the special case when \(\alpha_j = \alpha, j = 1,2, ..., n\). Then relation (9) takes the following form:

\[
x_j = \lambda^{\frac{1}{\alpha-1}} \left( \frac{1}{a_j \alpha} \right)^{\frac{1}{\alpha-1}}.
\]

At the same time

\[
\sum_{j=1}^{n} x_j = (\lambda)^{\frac{1}{\alpha-1}} \sum_{j=1}^{n} \left( \frac{1}{a_j \alpha} \right)^{\frac{1}{\alpha-1}} = C,
\]

whence

\[
(\lambda)^{\frac{1}{\alpha-1}} = \frac{C}{\sum_{j=1}^{n} \left( \frac{1}{a_j \alpha} \right)^{\frac{1}{\alpha-1}}}
\]

and

9
\[ x_j = \frac{c \left( \frac{1}{a_j^\alpha} \right)^{\frac{1}{\alpha - 1}}}{\frac{1}{a_j^\alpha}} \sum_{j=1}^{n} \left( \frac{1}{a_j^\alpha} \right)^{\frac{1}{\alpha - 1}}, \quad j = 1, 2, \ldots, n. \] (10)

Let us reduce (10) to a form more convenient for calculation:

\[ x_j = C \sum_{j=1}^{n} \left( \frac{a_j^{\alpha}}{(a_j^\alpha)^{\frac{1}{\alpha - 1}}} \right)^{\frac{1}{\alpha - 1}} = C \sum_{j=1}^{n} \frac{a_j^{\frac{1}{\alpha - 1}}}{a_j^{\frac{1}{\alpha - 1}}}, \]

which (in view \( a_j = \alpha \)) coincides with (8).

Let us now consider the problem of two-dimensional resource distribution with one-parameter production functions of system elements. Let us introduce the production function of the system elements

\[ f_j(x_1, x_2) = a_{1j}x_1^\alpha + a_{2j}x_2^\alpha; \quad j = 1, 2, \ldots, n; \] (11)

\[ F(x_1, x_2) = \sum_{j=1}^{n} f_j(x_1, x_2) = \sum_{j=1}^{n} (a_{1j}x_1^\alpha + a_{2j}x_2^\alpha). \] (12)

The restrictions on a two-dimensional resource have the following form:

\[ \sum_{j=1}^{n} x_{1j} = C_1, \] (13)

\[ \sum_{j=1}^{n} x_{2j} = C_2. \] (14)

Next

\[ \phi(x_1, x_2) = \sum_{j=1}^{n} (a_{1j}x_1^\alpha + a_{2j}x_2^\alpha) - \lambda_1 \left( \sum_{j=1}^{n} x_{1j} - C_1 \right) - \lambda_2 \left( \sum_{j=1}^{n} x_{2j} - C_2 \right), \]
\[ \frac{\partial \phi(x_1 x_2)}{\partial x_{1j}} = \alpha a_{1j} x_1^{\alpha-1} - \lambda_1 = 0, \quad j = 1, 2, \ldots, n, \quad (15) \]

\[ \alpha a_{1j} x_1^{\alpha-1} = \lambda_1, \quad x_1 = \left( \frac{\lambda_1}{\alpha a_{1j}} \right)^{\frac{1}{\alpha-1}} = \left( \frac{\lambda}{\alpha} \right)^{\frac{1}{\alpha-1}} \left( \frac{1}{a_{1j}} \right)^{\frac{1}{\alpha-1}}. \quad (16) \]

Let us find \( \lambda_1 \), by means of using (13)

\[ \sum_{j=1}^{n} x_{1j} = \left( \frac{\lambda}{\alpha} \right)^{\frac{1}{\alpha-1}} \sum_{j=1}^{n} \left( \frac{1}{a_{1j}} \right)^{\frac{1}{\alpha-1}} = C_1, \quad (17) \]

whence

\[ \left( \frac{\lambda}{\alpha} \right)^{\frac{1}{\alpha-1}} = \sum_{j=1}^{n} \left( \frac{1}{a_{1j}} \right)^{\frac{1}{\alpha-1}}, \quad (18) \]

\[ x_{1j} = \frac{C_1}{\sum_{j=1}^{n} \left( \frac{1}{a_{1j}} \right)^{\frac{1}{\alpha-1}}} = \frac{C_4 a_{1j}^{1-\alpha}}{\sum_{j=1}^{n} (a_{1j})^{1-\alpha}}, \quad (19) \]

\( j = 1, 2, \ldots, n. \)

Repeating actions (15)-(18) for \( x_{2j} \), we obtain a similar result:

\[ x_{2j} = \frac{C_2 a_{2j}^{1-\alpha}}{\sum_{j=1}^{n} (a_{2j})^{1-\alpha}}, \quad j = 1, 2, \ldots, n. \]

**Conclusions.** A method for solving the problem of rational distribution of a one-dimensional limited resource in the context of fuzzy initial data is proposed and justified. The technology and computational scheme for
implementing the method do not depend on the type of objective function of the problem and on the nature of the membership functions of its fuzzy parameters.

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